Reduction of the Mean Hedging Transaction Costs

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Abstract

Transaction costs of derivative hedging appear in financial markets. This paper considers the problem of delta hedging and the reduction of expected proportional transaction costs. In the literature the expected approximate proportional transaction costs are customarily estimated by the gamma term, usually the largest term of the associated series expansion. However, when options are to expire in a month or few weeks, other terms may become even larger so that more precise estimates are needed. In this paper, different higher-order estimates of proportional transaction costs are analyzed. The problem of the reduction of expected transaction costs is considered. As a result, a suitably adjusted delta is given, for which the expected approximate proportional transaction costs can be reduced. The order of the mean and the variance of the hedging error can be preserved. Several examples are provided.

Keywords: derivatives, delta hedging, transaction costs, hedging error

Introduction

In order to reduce the risk of highly leveraged derivative contracts, different hedging strategies can be applied. As known, the discrete-time delta hedging is a dynamic hedging technique widely used in practice. Transaction costs due to discrete-time delta hedging are highly dependent on the frequency of hedging and thus on the time length Δt between successive adjustments of the portfolio. If the hedging is relatively frequent, then the time Δt is relatively small. More frequent hedging means more precise hedging (smaller hedging error) as well as higher total transaction costs (see, for example, Boyle & Emanuel, 1980; Toft, 1996). Less frequent hedging means lower total transaction costs, but also higher hedging error.

This paper considers the problem of the reduction of the expected transaction costs for the case when the frequency of hedging is not necessarily lowered. Specifically, let the option value $V = V(t,S)$ be a function of the time t and the underlying assets price S. Suppose that the price $S = S(t)$ has lognormal distribution. In the continuous-time Black-Scholes model, where the hedging is instantaneous and the replication is perfect, the number of shares at time is given exactly by the delta—the current value of the partial derivative $V_S(t,S)$, where $V(t,S)$ is the solution of the Black-Scholes-Merton (BSM) equation (Black & Scholes, 1973; Merton, 1973). When the hedging is in discrete time, then over the time interval $(t, t + \Delta t)$ the number of shares is kept constant while at the time point $t + \Delta t$ the number of shares is readjusted to the new value $V_S(t + \Delta t, S + \Delta S)$. For details, see (Boyle and Emanuel, 1980).

The proportional transaction costs depend on the difference $|V_S(t + \Delta t, S + \Delta S) - V_S(t,S)|$, which is usually approximated by the gamma term—in most cases, the largest term of the associated Taylor series expansion (see, for example, Leland, 1985; Mastinsek, 2006; Toft, 1996). However, when options are near expiry, other terms of the series expansion are not necessarily small compared to the gamma term. Actually they can be even higher; thus, they cannot be ignored. This motivates further research. The following analysis will treat the problem more closely.

In order to deal with the subject, more precise estimates of proportional transaction costs will be considered. Consequently the problem of the reduction of the expected transaction costs will be analyzed. As a result, a suitably adjusted delta will be given for which the expected approximate proportional transaction costs can be reduced, while the order of the mean and the variance of the hedging error can simultaneously be preserved.

The paper is organized as follows: In the first section, the problem of proportional transaction costs and its reduction are considered. In the second section, the associated problem of the hedging error is studied. For illustration, an example of the European call option and several numerical results are given.

Transaction Costs

Let the number of shares N' at point $t + \Delta t$ be equal to the Black-Scholes delta: $N' = V_S(t + \Delta t, S + \Delta S)$, which is the hedge ratio customarily used in practice (compare Remark 1 below). If N is given by $N = V_S(t,S)$, then the proportional transaction costs C_{TR} at the rehedging moment $t + \Delta t$ are equal to:

$$C_{TR} = \frac{k}{2}|N'-N|(S+\Delta S) =$$

$$= \frac{k}{2}|V_S(t+\Delta t, S+\Delta S) - V_S(t,S)|(S+\Delta S) \qquad (1.1)$$

where k represents the round-trip transaction costs measured as a fraction of the volume of transactions. For details on the approximate transaction costs, see Leland (1985).

If $S = S(t)$ follows the geometric Brownian motion, then over the non-infinitesimal interval of the length Δt, its change can be approximated by:

$$\Delta S = S(t+\Delta t) - S(t) \approx \sigma S Z \sqrt{\Delta t} + \mu S \Delta t \qquad (1.2)$$

where σ is volatility, μ is the drift rate, and Z is the normally distributed variable with mean zero and variance one; in short $Z \sim N(0,1)$. For details, see Hull (2006). As noted, in this case, the first-order Taylor series approximation $|N'-N|$ of in (1.1) can be given by the partial derivative V_{SS} (the gamma), provided that other terms of the series are relatively small (Leland, 1985):

$$\Delta N = |N'-N| \approx |V_{SS}(t,S)\sigma S Z \sqrt{\Delta t}|, \qquad (1.3)$$

However, in many cases in practice, other partial derivatives of the series (like V_{St}) as well as the associated series terms may be too high to be neglected, as shown in example 1.

Example 1

Let V be the value of the European call option. Using the BSM formula (see (3.1) in the Appendix), the following ratio q between the partial derivatives can be obtained:

$$q := \frac{V_{St}}{V_{SS}} = \frac{S}{2T}\left[\ln S\middle/S_0 - (\tfrac{1}{2}\sigma^2 + r)T\right]$$

where S_0 is the exercise price and T the time to expiry.

Suppose that $S = 110\$$, $S_0 = 100\$$, $\sigma = 0.2$, $r = 0.05$. Using the previous formula, very large ratios will be obtained:

if $T = 0.1$, then $q = 48.6$,
if $T = 0.05$, then $q = 101.0$,
if $T = 0.02$, then $q = 258.3$.

Moreover, if $\Delta S = 0.5$ and $\Delta t = 0.01$, then the gamma term is not necessarily the largest term of the associated approximating series (1.4). Thus, other terms of the approximating series cannot be neglected. In order to deal with the problem, the following higher-order estimate can be considered:

$$\Delta N = |N'-N| =$$

$$= \left|V_{SS}(t,S)\Delta S + V_{St}(t,S)\Delta t + \tfrac{1}{2}V_{SSS}(t,S)\Delta S^2 + O(\Delta t^{3/2})\right|, \qquad (1.4)$$

where $O(.)$ is the order of the error. Consequently, the problem of the reduction of expected proportional transaction costs can be treated.

The objective of this paper is to obtain an appropriate choice of such that the expected transaction costs can be reduced while the order of the mean and the variance of the hedging error can be preserved. In particular, let us consider the adjusted hedge ratio of the form:

$$N = V_S(t + \alpha \Delta t, S) \qquad 0 \le \alpha \le 1, \tag{1.5}$$

where the parameter α is arbitrary and the number of shares N' is equal to the Black-Scholes delta: $N' = V_S(t+ \Delta t, S + \Delta S)$. For details, see Remark 1 below. In this case, we have:

$$\Delta N = |N' - N| =$$

$$= \left| V_{SS}(t,S)\Delta S + (1-\alpha)V_{St}(t,S)\Delta t + \right.$$

$$\left. + \tfrac{1}{2} V_{SSS}(t,S)\Delta S^2 + O(\Delta t^{3/2}) \right| \tag{1.6}$$

For simplicity of exposition, let us assume that $\mu = 0$ (as proposed by Leland (1985) the drift term in (1.2) may be neglected, when Δt is small). Then ΔN in (1.6) can be approximated to the order $O(\Delta t^{3/2})$ in the following way:

$$\Delta N \approx D = \left| V_{SS}(t,S)\sigma S \sqrt{\Delta t} Z + (1-\alpha)V_{St}(t,S)\Delta t + \right.$$

$$\left. + \tfrac{1}{2} V_{SSS}(t,S)(\sigma S Z \sqrt{\Delta t})^2 \right|$$

We rewrite D briefly:

$$D = b\left| aZ + (1-\alpha)c + Z^2 \right|, \tag{1.7}$$

where

$$b = \tfrac{1}{2} V_{SSS}(t,S)(\sigma S \sqrt{\Delta t})^2,$$

$$a = \frac{V_{SS}(t,S)\sigma S \sqrt{\Delta t}}{\tfrac{1}{2} V_{SSS}(t,S)(\sigma S \sqrt{\Delta t})^2},$$

$$c = \frac{V_{St}(t,S)\Delta t}{\tfrac{1}{2} V_{SSS}(t,S)(\sigma S \sqrt{\Delta t})^2}. \tag{1.8}$$

The parameters a, b, c depend on S, σ, Δt and the time to expiry T.

Remark 1

At time $t + \Delta t$, when the option is near the expiration date and the stock price $S + \Delta S$ is known, the hedger may choose a different hedging frequency using, for instance, price-based or delta-based rebalancing. Then the adjusted delta $V_S(t + \Delta t + \beta \Delta t', S + \Delta S)$ $\beta \ne 0$ can be calculated with respect to the new stock price $S + \Delta S$. Alternatively, the two-period model can be considered. However, in this case, the terms of the form $|aZ + (1 + \beta - \alpha)c + Z^2|$ will appear; thus, the optimization problem with the two unknown parameters has to be treated.

Remark 2

In practical cases where Δt is relatively small, the V_{SS} term in (1.6) is usually larger than the V_{SSS} the term so that $|\alpha|$ given by (1.8) is larger than 1. In particular, for the European call option, specific values are given in the Appendix (formula (3.2) and Example 8); moreover, parameter c is in most practical cases negative. The explicit formula c for is given by (3.3) in the Appendix. Therefore, the following problem associated with the reduction of the expected proportional transaction costs C_{TR} given by (1.1) can be considered:

Proposition 1

If $a>1$ and $c<0$, then the minimal value

$$\min_{\alpha} E\left| aZ + (1-\alpha)c + Z^2 \right| \tag{1.9}$$

is obtained for an α that satisfies the estimates:

$$1 - \omega_1 < \alpha < 1 - \omega_2 \tag{1.10}$$

where the constants ω_1, ω_2 depend on a, c and are given explicitly by the formulae (1.16) through (1.20), provided below.

Proof

If we introduce a new variable $Y = aZ + Z^2$, the minimization problem can be written as:

$$\min_{y} E\left| Y - y \right|. \tag{1.11}$$

As known from stochastic analysis, its solution is given by the *median* y_m of Y:

$$P(Y < y_m) = \frac{1}{2}. \tag{1.12}$$

The value $y_m > 0$ can be obtained from the cumulative normal distribution function $\Phi(z)$ of Z. Using (1.12), the following relationship holds:

$$P(z_1 < Z < z_2) = \Phi(z_2) - \Phi(z_1) = \frac{1}{2}, \tag{1.13}$$

where z_1, z_2 are solutions of the quadratic equation: $z^2 + az - y_m = 0$ and thus are given by:

$$z_{1,2} = \frac{1}{2}\left(-a \mp a \sqrt{1 + \frac{4y_m}{a^2}}\right).$$

Using the binomial (Taylor) series expansion, we have:

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots \qquad |x| < 1 \ . \qquad (1.14)$$

Hence, for $x = \frac{4y_m}{a^2} < 1$, we get the estimates:

$$-a - \frac{y_m}{a} < z_1 < -a \text{ and } 0 < z_2 < \frac{y_m}{a} \ . \qquad (1.15)$$

i) Using (1.13), (1.15), and the monotonicity of $\Phi(z)$, we find:

$$\Phi(z_1) < \Phi(-a) \text{ and } \Phi(z_2) = \frac{1}{2} + \Phi(z_1) < \frac{1}{2} + \Phi(-a) \ .$$

Hence,

$$z_2 < \Phi^{-1}(\frac{1}{2} + \Phi(-a)) =: z_a \qquad (1.16)$$

and based on the quadratic equation, it follows:

$$y_m < z_a^2 + az_a =: y_a \ . \qquad (1.17)$$

ii) Moreover, using (1.13) and (1.15), we also have:

$$-a - \frac{y_a}{a} < z_1$$

and

$$\frac{1}{2} + \Phi(-a - \frac{y_a}{a}) < \frac{1}{2} + \Phi(z_1) = \Phi(z_2) \ .$$

Based on the monotonicity of Φ, it follows:

$$z_b := \Phi^{-1}(\frac{1}{2} + \Phi(-a - \frac{y_a}{a})) < z_2 \qquad (1.18)$$

and

$$y_b := z_b^2 + az_b < y_m \ . \qquad (1.19)$$

Hence, using (1.17) and (1.19), it follows:

$$y_b < y_m = (1-\alpha)|c| < y_a$$

Thus, we get the estimates (1.10):

$$1 - \omega_1 < \alpha < 1 - \omega_2$$

where

$$\omega_1 := \frac{y_a}{|c|} \text{ and } \omega_2 := \frac{y_b}{|c|} \qquad (1.20)$$

As mentioned, in many practical cases where the option is not near expiry and Δt is small, the V_{SS} term in (1.6) is usually much larger than the V_{SSS} term, so that $|\alpha|$ is relatively large. This means that the value of $\Phi(-|a|)$ is very small. If the constants ω_1, ω_2 are very small, the optimal delta is close to the standard Black-Scholes delta. For illustration, let us give some examples.

Example 2

Let us assume that $4 < a < 20$ and $-1 < c < -0.6$. (For specific values a and c in the case of the European call option, see the Appendix.) Based on the tables of the cumulative normal distribution function $\Phi(z)$ of Z, we find:

$$\Phi(-a) < \Phi(-4) < 0.0001$$

$$\Phi(z_a) < 0.5001$$

Hence, using (1.16), $z_a < 0.00026$. Based on the assumption of a and using (1.17) and (1.20), we have:

$$y_a < 0.0053$$

and

$$\omega_1 = \frac{y_a}{|c|} < 0.01$$

Thus, based on (1.10), the optimal α satisfies the estimates:

$$0.99 < \alpha < 1 \ . \qquad (1.21)$$

Example 3

Suppose that $a > 1$ is not very large (e.g., $a = 2$) and $-1 \le c \le -0.6$. Then we find:

$$\Phi(-2) = 0.0228$$

and

$$\Phi(z_a) = 0.5228$$

Moreover, from the tables for Φ we find: $z_a < 0.058$. Based on (1.17), it follows:

$y_a < 0.12$ (1.22)

Using (1.18) we also get:

$0.04 < \Phi^{-1}(0.5197) = \Phi^{-1}(\frac{1}{2} + \Phi(-2 - \frac{0.12}{2})) < z_b$

Hence, based on (1.19), we have:

$0.082 < y_b$ (1.23)

Therefore, (1.22) and (1.23) lead to: $0.08 < y_m < 0.12$.

Using (1.20), it follows:

$1 - \frac{0.12}{|c|} < \alpha < 1 - \frac{0.08}{|c|}$

Thus, when $-1 \le c \le -0.6$, then the optimal α satisfies the estimates:

$0.80 < \alpha < 0.92$. (1.24)

For particular values of c, sharper estimates can be obtained. For instance,

 if $c = -0.6$, then $0.80 < \alpha < 0.87$

and

 if $c = -1$, then $0.88 < \alpha < 0.92$

As shown in (1.1), (1.6), and (1.7), proportional transaction costs C_{TR} can be approximated using $\Delta N \approx D$, where $D = b|aZ + (1-\alpha)c + Z^2|$. Let us illustrate the conclusions with the following numerical results.

Example 4

Let $a = 1.2$, and $c = -1$. Then by direct calculations of the expected value, we get the following results for different values of α:

Table 1

$\alpha =$	0.	0.3	0.5	0.8	0.9	1.
$\Delta N \approx$	1.296b	1.208b	1.172b	1.160b	1.169b	1.188b

This shows that the expected approximate proportional transaction costs C_{TR} for the standard delta ($\alpha = 0$) are approximately 12% higher than those where the adjusted delta ($\alpha = 0.8$) is used. Thus, using the appropriate delta, they can be reduced by 10.5%.

Example 5

Let $a = 2$, and $c = -1$. In this case, we get the following results for different values of α:

Table 2

$\alpha =$	0.	0.3	0.5	0.8	0.9	1.
$\Delta N \approx$	1.786b	1.707b	1.670b	1.641b	1.639b	1.642b

In this case, the expected approximate proportional transaction costs C_{TR} for the standard delta ($\alpha = 0$) are approximately 9% higher than those where the adjusted delta ($\alpha = 0.9$) is used.

Remark 3

For $a < -1$, the proof and the estimates can be given in a similar way as for $a > 1$. In this case, the symmetry of the Gaussian (density) function and the symmetry between $(z^2 - az)$ and $(z^2 + az)$ can be used. Thus, using an analogous argument, we can give here explicit estimates as well. The following result can be obtained.

Proposition 2

If $a < -1$ and $c < 0$, then the minimal value $\min_{\alpha} E|aZ + (1-\alpha)c + Z^2|$ is obtained for an α that satisfies the estimates: $1 - \omega_1 < \alpha < 1 - \omega_2$, where constants ω_1, ω_2 depend on a and c and are given by formulae (1.26) through (1.30) below.

Proof:

In this case, using (1.14), we generate the following estimates:

$-\frac{y_m}{|a|} < z_1 < 0$ and $|a| < z_2 < |a| + \frac{y_m}{|a|}$ (1.25)

i) First, based on the monotonicity of Φ, we have:

$\Phi(|a|) < \Phi(z_2)$,

$w_a := \Phi^{-1}(\Phi(|a|) - \frac{1}{2}) < z_1 < 0$. (1.26)

Hence, $|z_1| < |w_a|$, and given that $a < 0$, it follows:

$y_m < w_a^2 - aw_a = w_a^2 + |aw_a| =: y_a$ (1.27)

ii) Moreover, using (1.25), we get:

$$\Phi(z_2) < \Phi(|a| + y_a/|a|)$$

$$\Phi(z_1) < \Phi(|a| + y_a/|a|) - \frac{1}{2}$$

Thus,

$$z_1 < \Phi^{-1}\left[\Phi(|a| + y_a/|a|) - \frac{1}{2}\right] =: w_b \qquad (1.28)$$

Using the quadratic equation and based on $|z_1| > |w_b|$, we also get:

$$y_m > w_b^2 + |aw_b| =: y_b \qquad (1.29)$$

Hence, based on (1.27) and (1.29), we have $y_b < y_m < y_a$. Thus, it follows that $1 - \omega_1 < \alpha < 1 - \omega_2$ where

$$\omega_1 := \frac{y_a}{|c|} \text{ and } \omega_2 := \frac{y_b}{|c|}. \qquad (1.30)$$

Example 6

Let us assume that $a = -2$ and $-1 \le c \le -0.6$.

i) Based on the tables of the cumulative normal distribution function $\Phi(z)$ of Z, we find:

$$\Phi(|a|) = \Phi(2) < 0.9772 < \Phi(z_2)$$

Hence, using (1.26), it follows:

$$\Phi(w_a) = 0.4772 < \Phi(z_1)$$

$$-0.058 < w_a < z_1 < 0a$$

and

$$|z_1| < |w_a| < 0.058$$

Using (1.27), we get:

$$y_m < w_a^2 + |aw_a| =: y_a < 0.12 \qquad (1.31)$$

ii) Based on (1.28), we have:

$$\Phi(z_2) < \Phi(2.06) = 0.9803$$

$$z_1 < \Phi^{-1}[0.4803] =: w_b < -0.04$$

Using (1.29) we get $0.08 < y_b < y_m$. Thus, based on (1.31):

$$0.08 < y_m < 0.12$$

Hence, using (1.30), it follows:

$$1 - \frac{0.12}{|c|} < \alpha < 1 - \frac{0.08}{|c|} \qquad (1.32)$$

Therefore, when $-1 \le c \le -0.6$, we have estimates for the optimal α : $0.8 < \alpha < 0.92$. In particular cases, (1.32) can be used to obtain the following sharper estimates:

if $c = -0.6$, then $0.80 < \alpha < 0.87$

and

if $c = -1$, then $0.88 < \alpha < 0.92$

Example 7

Let $a = -1.1$ *and* $c = -1$. Then using direct calculations of the expected value, we get the following results for different values of α:

Table 3

$\alpha =$	0.	0.3	0.5	0.8	0.9	1.
$\Delta N \approx$	1.246b	1.158b	1.123b	1.115b	1.127b	1.149b

This shows that the expected proportional transaction costs C_{TR} for the standard delta ($\alpha = 0$) are again approximately 12% higher than those where the adjusted delta ($\alpha = 0.8$) is taken.

Next, let us analyze the hedging error when—instead of the standard delta—the adjusted delta is used. We will show that, in this case, the order of the mean and the variance of the hedging error can be preserved.

Hedging Error

As above, let us assume that $S = S(t)$ following the geometric Brownian motion:

$$dS(t) = \mu S dt + \sigma S dW, \qquad (2.1)$$

where μ is the expected annual drift rate, σ the volatility, and $W(.)$ the Brownian motion. Thus, over the interval of length Δt, the stock price change can be given by:

$$S(t + \Delta t) = S(t) \exp((\mu - \tfrac{1}{2}\sigma^2)\Delta t + \sigma Z\sqrt{\Delta t}), \qquad (2.2)$$

where Z ~ N(0,1). For details, see Hull (2006). Then the change $\Delta S = S(t + \Delta t) - S(t)$ of $S = S(t)$ over the interval of length Δt can be approximated using the Taylor series (see Mastinsek, 2012):

$$\Delta S =$$
$$= S\left[\sigma Z\sqrt{\Delta t} + (\mu - \tfrac{1}{2}\sigma^2)\Delta t + \tfrac{1}{2}\sigma^2 Z^2 \Delta t + \sigma(\mu - \tfrac{1}{2}\sigma^2)Z\Delta t^{3/2}\right] +$$
$$+ O(\Delta t^2)$$
$$(2.3)$$

With respect to (1.2), in this way, higher-order estimates of the hedging error ($O(\Delta t^2)$) can be given. As usual, let us assume that at time t a portfolio consists of a long position in the option and a short position in $N(t)$ units of stock S, so that the portfolio Π value at time t is equal to:

$$\Pi = V - N(t)S \qquad (2.4)$$

The return of the portfolio value over the interval $[t, t + \Delta t]$ is then equal to

$$\Delta\Pi = \Delta V - N(t)\Delta S \qquad (2.5)$$

as the number of shares $N(t)$ is held fixed during the time step Δt. The change ΔV of the option value $V(t,s)$ over the time interval of length Δt is, based on the Taylor series expansion, equal to:

$$\Delta V = V(t + \Delta t, S + \Delta S) - V(t, S) = V_t(t, S)\Delta t + V_S(t, S)\Delta S +$$
$$+ V_{St}(t, S)\Delta t \Delta S + \tfrac{1}{2}V_{SS}(t, S)(\Delta S)^2 + \tfrac{1}{6}V_{SSS}(t, S)(\Delta S)^3 + O(\Delta t^2).$$
$$(2.6)$$

Thus, based on (2.4), (2.5), and (2.6), the change of the portfolio value is equal to:

$$\Delta\Pi = V_t(t, S)\Delta t + (V_S(t, S) - N(t))\Delta S + V_{St}(t, S)\sigma S Z\Delta t^{\frac{3}{2}} +$$
$$+ \tfrac{1}{2}V_{SS}(t, S)S^2\left[\sigma^2 Z^2 \Delta t + 2\sigma\left[(\mu - \tfrac{1}{2}\sigma^2) + \tfrac{1}{2}\sigma^2 Z^2\right]Z\Delta t^{3/2}\right] +$$
$$+ \tfrac{1}{6}V_{SSS}(t, S)\sigma^3 S^3 Z^3 \Delta t^{\frac{3}{2}} + O(\Delta t^2).$$
$$(2.7)$$

If the amount Π is invested in a riskless asset (e.g., bonds) with an interest rate r, then over the interval of length Δt the return to the riskless investment is equal to:

$$\Delta B = \Pi \exp(r\Delta t) - \Pi = \Pi r\Delta t + O(\Delta t^2). \qquad (2.8)$$

In this case, the hedging error ΔH, defined as the difference between the return $\Delta\Pi$ to the portfolio value and the return ΔB to the bond value, is equal to $\Delta H = \Delta\Pi - \Delta B$. Hence, based on (2.7) and (2.8), we get:

$$\Delta H = \Delta\Pi - \Delta B = V_t(t, S)\Delta t + (V_S(t, S) - N(t))\Delta S -$$
$$- (V - N(t)S)r\Delta t + V_{St}(t, S)\sigma S Z\Delta t^{\frac{3}{2}} +$$
$$+ \tfrac{1}{2}V_{SS}(t, S)S^2\left[\sigma^2 Z^2 \Delta t + 2\sigma\left[(\mu - \tfrac{1}{2}\sigma^2) + \tfrac{1}{2}\sigma^2 Z^2\right]Z\Delta t^{3/2}\right] +$$
$$+ \tfrac{1}{6}V_{SSS}(t, S)\sigma^3 S^3 Z^3 \Delta t^{\frac{3}{2}} + O(\Delta t^2).$$
$$(2.9)$$

Then the following result can be concluded.

Proposition 3

Let σ be the annualized volatility and r the annual interest rate of a riskless asset. Let $V(t,S)$ be the solution of the Black-Scholes-Merton equation:

$$V_t(t, S) + \tfrac{1}{2}\sigma^2 S^2 V_{SS}(t, S) + rS V_S(t, S) - rV(t, S) = 0. \qquad (2.10)$$

If the number of shares $N(t)$ held short over the rebalancing interval of length, Δt is equal to:

$$N(t) = V_S(t + \alpha\Delta t, S)$$

where

$$0 \le \alpha \le 1, \qquad (2.11)$$

then the mean and the variance of the hedging error are of order $O(\Delta t^2)$.

Proof

Let us sketch the proof (for details, see Mastinsek, 2012). Based on the assumption $N = V_S(t + \alpha\Delta t, S)$, it holds that $V_S(t + \alpha\Delta t, S) = V_S(t, S) + V_{St}(t, S)\alpha\Delta t + O(\Delta t^2)$. We put $N = V_S(t + \alpha\Delta t, S)$ into equation (2.9) and apply the BSM equation (2.10) to equation (2.9). Thus, the terms of equation (2.9) associated with the terms in (2.10) are cancelled, and it follows that:

$$\Delta H = (1-\alpha)V_{St}(t,S)\sigma S Z \Delta t^{\frac{3}{2}} +$$

$$+ \frac{1}{2}V_{SS}(t,S)S^2 \left[\sigma^2(Z^2-1)\Delta t + 2\sigma\left[(\mu - \tfrac{1}{2}\sigma^2) + \tfrac{1}{2}\sigma^2 Z^2\right]Z\Delta t^{3/2}\right] +$$

$$+ \frac{1}{6}V_{SSS}(t,S)\sigma^3 S^3 Z^3 \Delta t^{\frac{3}{2}} + O(\Delta t^2).$$

Based on the assumption $Z\sim N(0,1)$, $E(Z) = 0$, $E(Z^2) = 1$, and $E(Z^3) = 0$. Thus, it follows that the mean of the hedging error satisfies the equation $E(\Delta H) = 0 + O(\Delta t^2)$ for all α, $0 \leq \alpha \leq 1$. In light of this result and the fact that $E(Z^{2n}) = 1.3.5...(2n-1)$ and $E(Z^{2n-1}) = 0$, *for n = 1, 2, 3, ...,* then the variance of ΔH can be readily calculated.

Conclusion

In the preceding analysis, the problem of expected proportional transaction costs due to discrete-time delta hedging has been considered. A suitably adjusted delta associated with the frequency of hedging and the time sensitivity of the delta were given. In this way, expected approximate proportional transaction costs can be reduced while the order of the mean and the variance of the hedging error can be preserved.

Appendix

Let $V(t, S)$ denote the value of a European call option. Using the BSM formula, we get:

$$V_{SS}(t,S) = N'(d_1)\frac{1}{\sigma S \sqrt{T}} \; ,$$

$$V_{St}(t,S) = N'(d_1)\frac{\ln S/S_0 - (\tfrac{1}{2}\sigma^2 + r)T}{2\sigma T \sqrt{T}} \tag{3.1}$$

with

$$d_1 = \frac{\ln S/S_0 + (\tfrac{1}{2}\sigma^2 + r)T}{\sigma\sqrt{T}} \qquad N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Here S_0 is the strike price, σ the annual volatility, r the interest rate, and T the time to expiry. Moreover, $N(x)$ is the cumulative probability distribution function for a standardized normally distributed variable. For details, see Hull (2006). Based on the definition of *a* given in (1.8), we directly obtain:

$$a = \frac{-2\sqrt{T}}{\sqrt{\Delta t}(d_1 + \sigma\sqrt{T})} \; . \tag{3.2}$$

This means, when Δt is relatively small and d_1 is not too large (the option price is not too far from the strike price), $|a|$ is larger than 1.

Example 8

Suppose that $\sigma = 0.2$, $\Delta t = 0.01$ *and r = 0* . Thus, we have:

when T = 0.25, and $0.85 < \left|S/S_0\right| < 1.2$ *then* $|a| > 5.0$

when T = 0.1, and $0.85 < \left|S/S_0\right| < 1.2$ *then* $|a| > 1.9$

when T = 0.4, and $0.9 < \left|S/S_0\right| < 1.1$ *then* $|a| > 1.4$

When the option is relatively deep in or out of the money (for instance, if $\left|S/S_0\right| > 1.2$), the gamma and delta options change very little over time. Thus, the needed readjustments of the portfolio are small and the proportional transaction costs low.

Next let us consider the parameter *c* for the European call option. Note that the terms associated with V_{St}, V_{SSS} in (1.6) are of the same order so that *c* is independent of Δt. In that case, using the BSM formula, we have:

$$c = \frac{V_{St}(t,S)\Delta t}{\tfrac{1}{2}V_{SSS}(t,S)(\sigma S\sqrt{\Delta t})^2} = \frac{-d_1 + \frac{(\sigma^2 + 2r)}{\sigma}\sqrt{T}}{d_1 + \sigma\sqrt{T}} \; . \tag{3.3}$$

References

1. Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, *81*, 637–659. http://dx.doi.org/10.1086/260062

2. Boyle, P., & Emanuel, D. (1980). Discretely adjusted option hedges. *Journal of Financial Economics, 8*, 259–282. http://dx.doi.org/10.1016/0304-405X(80)90003-3

3. Hull, J. C. (2006). *Option, futures & other derivatives*. Pearson Prentice-Hall.

4. Leland, H. E. (1985). Option pricing and replication with transaction costs. *Journal of Finance, 40*, 1283–1301. http://dx.doi.org/10.1111/j.1540-6261.1985.tb02383.x

5. Mastinsek, M. (2006). Discrete-time delta hedging and the Black-Scholes model with transaction costs. *Mathematical Methods of Operations Research, 64*, 227–236. http://dx.doi.org/10.1007/s00186-006-0086-0
6. Mastinsek, M. (2012). Charm-adjusted delta and delta gamma hedging. *Journal of Derivatives, 19*(3), 69–76. http://dx.doi.org/10.3905/jod.2012.19.3.069
7. Merton, R. C. (1973). Theory of rational option pricing. *Bell Journal of Economics and Management Science, 4*, 141–183. http://dx.doi.org/10.2307/3003143
8. Toft, K. B. (1996). On the mean-variance tradeoff in option replication with transactions costs. *The Journal of Financial and Quantitative Analysis, 31*(2), 233–263. http://dx.doi.org/10.2307/2331181

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Redukcija povprečnih transakcijskih stroškov hedging tehnike

Izvleček

Na finančnih trgih se pri uporabi hedging tehnike pojavijo transakcijski stroški. V tem članku se obravnava problem uporabe delta hedging tehnike ter redukcije proporcionalnih transakcijskih stroškov. V literaturi navedene metode običajno temeljijo le na uporabi tako imenovanega faktorja gama, ki ponavadi predstavlja največji člen v aproksimacijski vrsti. Toda pri opcijah s kratkim časom dospetja, mesec ali nekaj tednov, lahko drugi členi vrste postanejo celo večji. Tedaj so potrebne natančnejše aproksimacije. V tem članku so analizirane aproksimacije višjega reda in njihova uporaba pri zmanjšanju povprečnih proporcionalnih transakcijskih stroškov. Na podlagi analize je podan ustrezno prilagojen faktor delta, s katerim se povprečni aproksimativni proporcionalni transakcijski stroški lahko zmanjšajo. Pripadajoča napaka hedging tehnike se pri tem ne poveča. Za ilustracijo metode je dodanih nekaj primerov.

Ključne besede: finančni derivati, transakcijski stroški, delta hedging tehnika